

# Hedging and the Greeks

Math 485

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## 1 Goal:

To introduce the idea of hedging a Euro-style derivative in continuous time and defining the Greeks in option pricing.

## 2 Motivation:

Suppose an option seller sells a Euro-style derivative that pays  $V_T = \phi(S_T)$  at time  $T$ . We already learned that he should charge  $V_0 = E(e^{-rT}\phi(S_T))$  for the option at time 0.

Now the question is what should the option seller should do with  $V_0$ ? He is obligated to pay out  $\phi(S_T)$  (For example,  $\phi(S_T) = (S_T - K)^+$  if the derivative is a Euro Call option) at time  $T$ . Certainly he cannot just invest  $V_0$  in the bank and hope that he will have enough money to cover the random amount  $\phi(S_T)$  that needs to be paid out at time  $T$ . Clearly he needs to invest  $V_0$  in a portfolio that is a combination of the stock  $S$  and the money market.

But how much should he hold in stocks? Recall from the binomial tree model, we learned that to hedge a Euro-style derivative, at any time  $k$  the option seller should hold  $\Delta_k := \frac{V_{k+1} - V_k}{S_{k+1} - S_k}$  shares of stock and put the rest of his money into the money market. Then at the expiration time  $n$ , the value of his portfolio will be exactly equal to  $V_n$ , the amount that needs to be paid out. We will apply this idea in continuous time as well. This is the idea of Delta hedging.

### 3 Delta hedging:

The idea: We divide the interval  $[0, T]$  into  $n$  subintervals, each with length  $\delta$  ( $\delta$  small). We denote each grid point of these subintervals by  $t_k, 0 = t_0 < t_1 < \dots < t_n = T$ .

We construct a self-financing portfolio that consists of the underlying stock and the money market as followed: At each time  $t_k$ , we will hold  $\Delta_k := \frac{\partial V}{\partial S}(t_k)$  shares of stock. We claim that in this way, the value of the portfolio at time  $T$  will approximately be equal to the value of the derivative  $V_T = \phi(S_T)$ .

Reason: By Ito's formula

$$V_{t_{k+1}} - V_{t_k} \approx \left( \frac{\partial V}{\partial t}(t_k) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t_k) \sigma^2 S_{t_k}^2 \right) \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k}).$$

Since  $V_t$  satisfies the Black-Scholes PDE, we have

$$\frac{\partial V}{\partial t}(t_k) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t_k) \sigma^2 S_{t_k}^2 = -\frac{\partial V}{\partial S} r S(t_k) + r V(t_k).$$

Plug this in the above:

$$\begin{aligned} V_{t_{k+1}} - V_{t_k} &\approx \left( -\frac{\partial V}{\partial S} r S(t_k) + r V(t_k) \right) \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k}) \\ &= \left( V(t_k) - \frac{\partial V}{\partial S} S(t_k) \right) r \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k}). \end{aligned}$$

Now suppose at time  $t_k$  we have a portfolio  $\pi$  that satisfies  $\pi(t_k) \approx V(t_k)$ . We purchase  $\frac{\partial V}{\partial S}(t_k)$  shares of stock, which leaves us with  $\pi(t_k) - \frac{\partial V}{\partial S} S(t_k)$  to put into the bank. At time  $t_{k+1}$  the value of our portfolio is (because of self-financing)

$$\pi(t_{k+1}) = \pi(t_k) + \left( \pi(t_k) - \frac{\partial V}{\partial S} S(t_k) \right) r \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k})$$

Note that we're in discrete time so the growth in 1 period of time of the money market portion is the interest rate times the length of that period, which is  $\delta$ .

But since  $\pi(t_k) \approx V(t_k)$  we have

$$\begin{aligned} \pi(t_{k+1}) &\approx V(t_k) + \left( V(t_k) - \frac{\partial V}{\partial S} S(t_k) \right) r \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k}) \\ &\approx V(t_{k+1}). \end{aligned}$$

So the approximation extends to the next period. The quantity  $\frac{\partial V}{\partial S}$ , the first partial derivative of  $V$  with respect to  $S$ , is thus seen to be very important in hedging, and it's called the Delta, in symbol  $\Delta$ , the first Greek we encounter in this section.

## 4 Computing $\frac{\partial V}{\partial S}$ :

The above derivation is valid for any Euro-style derivative. However, the relevant question is: how much is exactly  $\frac{\partial V}{\partial S}$ ? Or how to compute the Delta of a certain Euro derivative? This is difficult in general and usually one needs to use numerical techniques. However, when we specialize to certain cases of  $V_T = \phi(S_T)$ , for example  $\phi(S_T) = S_T^k$  for some integer  $k$  then explicit computation of the Delta is possible. In this section we show how to compute the Delta of the most important derivative we encounter in this class: the Euro-Call option.

Recall that the Black-Scholes formula gives for a Euro call that pays  $(S_T - K)^+$  at time  $T$ :

$$\begin{aligned} V(t, S_t) &= S_t N(d_1(t, S_t)) - K e^{-r(T-t)} N(d_2(t, S_t)), \\ d_1(t, S_t) &= \frac{(r + \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}} \\ d_2(t, S_t) &= \frac{(r - \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}} \end{aligned}$$

It is also easy to see that

$$\frac{\partial}{\partial S} d_2(t, S_t) = \frac{\partial}{\partial S} d_1(t, S_t) = \frac{1}{S_t \sigma \sqrt{T-t}}.$$

Therefore,

$$\frac{\partial V}{\partial S}(t) = N(d_1(t, S_t)) + S_t \phi_Z(d_1(t, S_t)) \frac{1}{S_t \sigma \sqrt{T-t}} - K e^{-r(T-t)} \phi_Z(d_2(t, S_t)) \frac{1}{S_t \sigma \sqrt{T-t}}.$$

We claim that

$$\phi_Z(d_1(t, S_t)) = K e^{-r(T-t)} \phi_Z(d_2(t, S_t)) \frac{1}{S_t}.$$

To see this, note that  $d_1(t, S_t) = d_2(t, S_t) + \sigma\sqrt{T-t}$ . Therefore,

$$\begin{aligned} \phi_Z(d_1) &= \phi_Z(d_2 + \sigma\sqrt{T-t}) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_2 + \sigma\sqrt{T-t})^2}{2}\right) \\ &= \phi_Z(d_2) \exp\left(\frac{-2d_2\sigma\sqrt{T-t} - \sigma^2(T-t)}{2}\right). \end{aligned}$$

One can check that

$$2d_2(t, S_t)\sigma\sqrt{T-t} + \sigma^2(T-t) = 2(r(T-t) - \log(K) + \log(S_t)).$$

Plug this into the above expression, the claim is checked. Thus we see a surprisingly simple result:  $\frac{\partial V}{\partial S}(t) = N(d_1(t, S_t))$ .

## 5 Predicting the future price of Euro Call option

So we see that the partial derivative of  $V$  with respect to  $S$  plays an important role in hedging. Indeed the Greeks are just various partial derivatives of  $V$  with respect to different parameters in the Black-Scholes model:  $t, r, \sigma, T, S$ . Some of them show up more often than others. In particular, two more Greeks that are important for our purpose are the ones that appear in Ito's formula:

$$\begin{aligned}\Theta(t) &:= \frac{\partial V}{\partial t}(t) \\ \Gamma(t) &:= \frac{\partial^2 V}{\partial S^2}(t),\end{aligned}$$

and of course previously we have

$$\Delta(t) := \frac{\partial V}{\partial S}(t).$$

Note that in this way the Greeks are random processes. They are functions of  $t$  and  $S_t$ . Their use is to measure the sensitivity of the option price with respect to the change of other parameters in the model. Again in general it may be difficult to compute the  $\Theta, \Gamma$  of a general derivative. But if we specialize to certain form of  $\phi(S_T)$  then the computation can be doable. In particular, for the Euro-Call option:

$$\begin{aligned}\Gamma(t) &= \frac{1}{\sigma S_t \sqrt{2\pi(T-t)}} e^{-\frac{d_1^2(t, S_t)}{2}} \\ \Theta(t) &= -r e^{-r(T-t)} K N(d_2(t, S_t)) - \frac{1}{2} \sigma^2 S_t^2 \Gamma(t).\end{aligned}$$

The formulas are complicated, but they are explicit and one can compute these quantities provided  $S_t, \sigma, r, T$  are given. Also at time  $t$ , using Black-Scholes formula we also know  $V_t$ . Therefore, Ito's formula gives for a small change in time  $t + \delta$

$$V_{t+\delta} \approx V_t + (\Theta(t) + \frac{1}{2} \Gamma(t) \sigma^2 S^2(t)) \delta + \Delta(t) (S_{t+\delta} - S_t).$$

Note: The book used  $V_{new}$  for  $V_{t+\delta}$  and only consider the case  $t = 0$ . So their formula is simpler than ours and our formula is slightly more general.